Solution to Assignment 5

8. We have
$$\left|\int_a^b f\right| \le \int_a^b |f| \le \int_a^b M = M(b-a)\ .$$
Note that the first inequality comes from the Riemann sums after passing to limit. In the next step we integrate a constant function, see Example 2.1 in Notes 2.

11. Suppose $\lim_{n\to\infty} S(f,\dot{\mathcal{P}}_n) > \lim_{n\to\infty} S(f,\dot{\mathcal{Q}}_n)$. Then we have
$$\begin{aligned}\overline{S}(f) &= \lim_{n\to\infty} \overline{S}(f,\mathcal{P}_n) \ge \lim_{n\to\infty} S(f,\dot{\mathcal{P}}_n)\\ &> \lim_{n\to\infty} S(f,\dot{\mathcal{Q}}_n) \ge \lim_{n\to\infty} \underline{S}(f,\mathcal{Q}) = \underline{S}(f)\end{aligned}$$
$\therefore\ \overline{S}(f) \ne \underline{S}(f)$, $f \notin \mathcal{R}[a,b]$, by Integrability Criterion I.

14. (a) $\frac{1}{3}(x_{i-1}^2 + x_{i-1}x_{i-1} + x_{i-1}^2) \le q_i^2 = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2) \le \frac{1}{3}(x_i^2 + x_i x_i + x_i^2)$
$\Rightarrow \quad 0 \le x_{i-1}^2 \le q_i^2 \le x_i^2 \quad \Rightarrow \quad 0 \le x_{i-1} \le q_i \le x_i$.

(b) $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)(x_i - x_{i-1}) = \frac{1}{3}(x_i^3 - x_{i-1}^3)$.

(c) Here we let $\dot{P}$ be the partition P with tags q_j. Then
$$S(Q;\dot{P}) = \sum_{i=1}^n Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}\sum_{i=1}^n (x_i^3 - x_{i-1}^3) = \frac{1}{3}(b^3 - a^3)\ .$$

(d) The function $x \mapsto x^2$ is integrable by Theorem 2.8(b) being the product of the linear function $x \mapsto x$ (Example 2.4 in Notes 2). Take $\dot{P}_n$ be tagged partitions whose length tending to 0. By letting $n \to \infty$, we see from (c) and Theorem 2.6 that
$$\int_a^b Q = \frac{1}{3}(b^3 - a^3)\ .$$
Note. By choosing the tag points z_j carefully, we can use the same method to evaluate the integral of x^n for all positive powers. You are encouraged to work it out for $n = 3$. After this effort, it is easy to guess which tags to choose in the general case.

15. Let $P = \{I_j := [x_{j-1}, x_j]\}_{j=1}^n$ be a partition of f on $[a,b]$.
Clearly, $\forall\ j$, $\sup_{I_j} f = \sup_{I_j+c} g$, $\inf_{I_j} f = \inf_{I_j+c} g$. Hence $\overline{S}(f,P) = \overline{S}(g,Q)$, $\underline{S}(f,P) = \underline{S}(g,Q)$, where $Q := \{I_j + c = [x_{j-1}+c, x_j+c]\}_{j=1}^n$ is a partition of g on $[a+c, b+c]$. It is now clear that $\overline{S}(g) = \overline{S}(f)$ and $\underline{S}(g) = \underline{S}(f)$, so by the first criterion, g is integrable and
$$\int_{a+c}^{b+c} g = \overline{S}(g) = \overline{S}(f) = \int_a^b f\ .$$
Note: This property is called the translation invariance of the Riemann integral.

Supplementary Exercises

Use the knowledge in Section 1, Notes 2.

1. Let f be a continuous function on (a,b) satisfying

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(f(x)+f(y)\Big), \quad \forall x,y \in (a,b).$$

Show that f is convex. Suggestion: Show

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{f(x_1)+\cdots+f(x_n)}{n} \ ,$$

for $n = 2^m$.

Solution. Let us show it holds for $n = 2^m$ first. Use induction on m. When $m = 1$, done by assumption. Assuming it holds at m, we show it for $m+1$. For $x_1, \cdots, x_{2^{m+1}}$, we have

$$\frac{x_1+\cdots+x_{2^{m+1}}}{2^{m+1}} = \frac{1}{2}\frac{x_1+\cdots+x_{2^m}}{2^m} + \frac{1}{2}\frac{x_{2^m+1}+\cdots+x_{2^{m+1}}}{2^m} \ .$$

Therefore, first by assumption and then by induction hypothesis

$$\begin{aligned} f\left(\frac{x_1+\cdots+x_{2^{m+1}}}{2^{m+1}}\right) &\le \frac{1}{2}f\left(\frac{x_1+\cdots+x_{2^m}}{2^m}\right) + \frac{1}{2}f\left(\frac{x_{2^m+1}+\cdots+x_{2^{m+1}}}{2^m}\right) \\ &\le \frac{1}{2}\left(\frac{f(x_1)+\cdots+f(x_{2^m})}{2^m} + \frac{f(x_{2^m+1})\cdots+f(x_{2^{m+1}})}{2^m}\right) \\ &= \frac{f(x_1)+\cdots+f(x_{2^{m+1}})}{2^{m+1}} \ . \end{aligned}$$

After we have proved the inequality for 2^m, we "collapse" it by taking $x = x_1 = \cdots = x_n$ and $y = x_{n+1} = \cdots = x_{2^m}$ to get

$$f\left(\frac{n}{2^m}x + \left(1-\frac{n}{2^m}\right)y\right) \le \frac{n}{2^m}f(x) + \left(1-\frac{n}{2^m}\right)f(y) \ ,$$

so the inequality holds for all λ of the form $n/2^m, 0 \le n \le 2^m$. Since every $\lambda \in (0,1)$ can be approximated by such rational numbers, by the continuity of f we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \ ,$$

so f is convex.

We point out that there exist discontinuous functions satisfying this "mean convex property" but is not convex. Google for it in case you are interested in such pathological example.

2. Let f be differentiable on $[a,b]$. Show that it is convex if and only if

$$f(y) - f(x) \ge f'(x)(y-x), \qquad \forall x,y \in [a,b].$$

What is the geometric meaning of this inequality?

Solution. Suppose f is convex, then by Theorem 1.5 of Notes 1, we have f' is increasing function. Let $x \neq y \in [a,b]$. By Mean-Value Theorem, $\exists \xi$ in between x and y such that

$$f(y) - f(x) = f'(\xi)(y-x).$$

Hence

$$\frac{f(y)-f(x)}{y-x} = f'(\xi) = \begin{cases} \geq f(x), & \text{if } x < y. \\ \leq f(x), & \text{if } x > y. \end{cases}$$

Suppose $f(y) - f(x) \geq f'(x)(y-x), \quad \forall x, y \in [a,b]$. We attempt to show that f' is increasing. Let $y > x$, by our assumption, we have

$$f(y) - f(x) \geq f'(x)(y-x)$$

and

$$f(x) - f(y) \geq f'(y)(x-y)$$

which imply

$$f'(y) \geq \frac{f(x)-f(y)}{x-y} = \frac{f(y)-f(x)}{y-x} \geq f'(x).$$

Therefore, f' is increasing. Again by Theorem 1.5 of Notes 1, f is convex on $[a,b]$.

The geometric meaning is, a differentiable function is convex if and only if its tangent line at any point always lies below the graph of the function.

3. Establish the following two inequalities

   (a)
   $$\sin x + \sin y + \sin z \leq \frac{3\sqrt{3}}{2} .$$

   (b)
   $$\sin x \ \sin y \ \sin z \leq \frac{3\sqrt{3}}{8} .$$

   (c)
   $$\frac{1}{3}\left(\frac{1}{\sin x} + \frac{1}{\sin y} + \frac{1}{\sin z}\right) \geq \frac{2}{\sqrt{3}} .$$

   Here x, y, z are the three interior angles of a triangle.

   Solution.

   (a) The sine function is concave on $[0, \pi]$. Therefore,

   $$\sin\frac{\pi}{3} = \sin\frac{x+y+z}{3} \geq \frac{\sin x + \sin y + \sin z}{3} ,$$

   implies the first inequality.

   (b) Next, the function $\log\sin x$ is concave everywhere (actually its second derivative is equal to $-1/\sin x^2 < 0$.) Therefore,

   $$\frac{\log\sin x + \log\sin y + \log\sin z}{3} \leq \log\sin\left(\frac{x+y+z}{3}\right) = \log\sin\frac{\pi}{3} ,$$

   implies the second inequality.

(c) Use the concavity of the function $1/\sin x$.

Since these functions are strictly concave, the inequality signs are strict unless $x = y = z$. Using $x + y + z = \pi$, conclude that equality signs hold in these three inequalities if and only if $x = y = z = \pi/3$, that is, for an equilateral triangle.

4. Establish the inequality

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}, \quad a, b, c > 0 .$$

Hint: Use of one the functions in (1).

Solution. Take log of both sides and apply Jensen's Inequality to $x \log x$.

5. Let P be the partition $\{-1, -\frac{1}{2}, 0, \frac{1}{3}, 1\}$ of $[-1, 1]$. Define $f : [-1, 1]$ by

$$f(x) = \begin{cases} -x & \text{if } x \in [-1, 0], \\ -x + 1 & \text{if } x \in (0, 1]. \end{cases}$$

(a) Find the Darboux upper and lower sums for f. Explain why the Darboux upper sum is not a Riemann sum.

(b) Use the integrability criterion to show that f is integrable and find its integral.

Solution.

(a)
$$\begin{aligned}
\overline{S}(f, P) &= \sum_{j=1}^{4} \sup_{I_j} f \ \Delta x_j \\
&= \left(\sup_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2} - (-1)\right) + \left(\sup_{x\in[-1/2,0]} -x\right)\left(0 - \left(-\frac{1}{2}\right)\right) \\
&\qquad + \left(\sup_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3} - 0\right) + \left(\sup_{x\in[1/3,1]} -x+1\right)\left(1 - \frac{1}{3}\right) \\
&= (1)\left(-\frac{1}{2} - (-1)\right) + \left(\frac{1}{2}\right)\left(0 - \left(-\frac{1}{2}\right)\right) + (1)\left(\frac{1}{3} - 0\right) + \left(\frac{2}{3}\right)\left(1 - \frac{1}{3}\right) \\
&= \frac{55}{36}
\end{aligned}$$

$$\begin{aligned}
\underline{S}(f, P) &= \sum_{j=1}^{4} \inf_{I_j} f \ \Delta x_j \\
&= \left(\inf_{x\in[-1,-1/2]} -x\right)\left(-\frac{1}{2} - (-1)\right) + \left(\inf_{x\in[-1/2,0]} -x\right)\left(0 - \left(-\frac{1}{2}\right)\right) \\
&\qquad + \left(\inf_{x\in[0,1/3]} -x+1\right)\left(\frac{1}{3} - 0\right) + \left(\inf_{x\in[1/3,1]} -x+1\right)\left(1 - \frac{1}{3}\right) \\
&= \left(\frac{1}{2}\right)\left(-\frac{1}{2} - (-1)\right) + 0\left(0 - \left(-\frac{1}{2}\right)\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3} - 0\right) + 0\left(1 - \frac{1}{3}\right) \\
&= \frac{17}{36}
\end{aligned}$$

The Darboux upper sum is not a Riemann sum because $\sup_{[0,1/3]} f = 1$ but we can't find any tag $z \in [0, 1/3]$ so that $f(z) = 1$, because of the definition of f.

(b) Take $P_n := \{x_i := -1 + i/n\}_{i=0}^{2n}$, hence $\|P_n\| \to 0$.

Then $$\overline{S}(f) = \lim \overline{S}(f, P_n) = \lim \left(\sum_{i=1}^{n} (-x_{i-1})\ \Delta x_i + \sum_{i=n+1}^{2n} (-x_{i-1}+1)\ \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} (-x_{i-1})\ \Delta x_i + \sum_{i=n+1}^{2n} \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} \left(1 - \frac{i-1}{n}\right)\left(\frac{1}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{1}{n}\right) \right) = 2 - \lim \frac{1}{n^2} \sum_{i=1}^{2n} (i-1) + 1$$
$$= 3 - \lim \frac{1}{n^2} \frac{(0 + (2n-1))2n}{2} = 3 - \lim \frac{2n-1}{n} = 3 - 2 = 1$$

and $$\underline{S}(f) = \lim \underline{S}(f, \mathcal{P}_n) = \lim \left(\sum_{i=1}^{n} (-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n} (-x_i + 1)\ \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} (-x_i)\ \Delta x_i + \sum_{i=n+1}^{2n} \Delta x_i \right)$$
$$= \lim \left(\sum_{i=1}^{2n} \left(1 - \frac{i}{n}\right)\left(\frac{1}{n}\right) + \sum_{i=n+2}^{2n} \left(\frac{1}{n}\right) \right) = 2 - \lim \frac{1}{n^2} \sum_{i=1}^{2n} i + 1$$
$$= 3 - \lim \frac{1}{n^2} \frac{(1+2n)2n}{2} = 3 - \lim \frac{1+2n}{n} = 3 - 2 = 1$$

Hence $\overline{S}(f) = 1 = \underline{S}(f)$, by integrability criterion, $f \in \mathcal{R}[-1,1]$ and $\displaystyle\int_{-1}^{1} f = 1$

6. Prove Cauchy criterion for integrability: f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two tagged partitions $\dot{P}, \dot{Q}$ with length less than δ,
$$|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon,$$
holds. (This criterion is proved in the text; pretend that it is not there.)

Solution.

$\Rightarrow$) Since $f \in R[a,b]$, $\exists\ L$ s.t. $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$,
$$|\ S(f, \dot{P}) - L\ | < \frac{\varepsilon}{2}, \quad \forall\ \|P\| < \delta.$$
For another Q, $\|Q\| < \delta$, we have a similar inequality.
$$|\ S(f, \dot{P}) - S(f, \dot{Q})\ | \le |\ S(f, \dot{P}) - L\ | + |\ S(f, \dot{Q}) - L\ | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\Leftarrow$) Let $\varepsilon/2 > 0$ and choose $P = Q$ but different tags so that
$$\left| S(f, \dot{P}) - S(f, \ddot{P}) \right| < \frac{\varepsilon}{2}\ ,$$
and
$$\left| \overline{S}(f, P) - S(f, \dot{P}) \right| < \frac{\varepsilon}{4}, \quad \left| \underline{S}(f, P) - S(f, \ddot{P}) \right| < \frac{\varepsilon}{4}\ .$$
As a result,
$$\begin{aligned} \left| \overline{S}(f, P) - \underline{S}(f, P) \right| \ &\le\ \left| \overline{S}(f, P) - S(f, \dot{P}) \right| + \left| S(f, \dot{P}) - S(f, \ddot{P}) \right| + \left| \underline{S}(f, P) - S(f, \ddot{P}) \right| \\ &<\ \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon\ . \end{aligned}$$

Therefore,

$$0 \le \overline{S}(f) - \underline{S}(f) \le \overline{S}(f, P) - \underline{S}(f, P) \le \varepsilon \ .$$

Since ε can be arbitrarily small, we must have $0 = \overline{S}(f) - \underline{S}(f)$, so f is integrable by the First Integrability Criterion.